

BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS OF C^* -ALGEBRA-VALUED SYMBOL.*

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Abstract

Let us consider the set $S^A(\mathbb{R}^n)$ of rapidly decreasing functions $G : \mathbb{R}^n \rightarrow A$, where A is a separable C^* -algebra. We prove a version of the Calderón-Vaillancourt theorem for pseudodifferential operators acting on $S^A(\mathbb{R}^n)$ whose symbol is A -valued. Given a skew-symmetric matrix, J , we prove that a pseudodifferential operator that commutes with $G(x + JD)$, $G \in S^A(\mathbb{R}^n)$, is of the form $F(x - JD)$, for F a C^∞ -function with bounded derivatives of all orders.

1 Introduction.

Throughout this work, A denotes a separable C^* -algebra and $S^A(\mathbb{R}^n)$ denotes the A -valued Schwartz space of smooth and rapidly decreasing functions on \mathbb{R}^n . On $S^A(\mathbb{R}^n)$ we define the A -valued inner product

$$\langle f, g \rangle = \int f(x)^* g(x) dx,$$

whose associated norm we denote by $\|\cdot\|_2$, $\|f\|_2 = \|\langle f, f \rangle\|^{\frac{1}{2}}$.

The completion of $S^A(\mathbb{R}^n)$ with this norm is a Hilbert A -module that we denote by E . The set of all adjointable (and therefore bounded) operators on E we denote by $\mathcal{B}^*(E)$. Let $CB^\infty(\mathbb{R}^n, A)$ denote the set of C^∞ -functions with bounded derivatives of all orders.

In section 2, we see a generalization of the Calderón-Vaillancourt Theorem, [1], for a pseudodifferential operator, $a(x, D)$, whose symbol, a , is in $CB^\infty(\mathbb{R}^{2n}, A)$.

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Following the steps of Seiler's proof in [8] (in fact going back to Hwang, [4]), we can see that $a(x, D)$ is bounded on E . Note that Seiler's result needs that one works on a Hilbert space where we have that the Fourier transform is unitary in the usual sense. One of the advantages of working with this norm, $\|\cdot\|_2$, is that the Fourier transform becomes an "unitary" operator on the completion of $S^A(\mathbb{R}^n)$; *i.e.* it is a Hilbert-module adjointable operator on E , whose inverse is equal to its adjoint. This norm $\|\cdot\|_2$ allows for a proof of the Calderón-Vaillancourt Theorem for pseudodifferential operators whose symbols are A -valued functions.

We also prove that these operators are adjointable. So, we have $a(x, D) \in \mathcal{B}^*(E)$, $a \in CB^\infty(\mathbb{R}^{2n}, A)$.

In [7], Rieffel defines a deformed product in $CB^\infty(\mathbb{R}^n, A)$, depending on an anti-symmetric matrix, J , by

$$F \times_J G(x) = \int (2\pi)^{-n} e^{iu \cdot v} F(x + Ju) G(x + v) dudv.$$

It is not difficult to see that the left-regular representation of $CB^\infty(\mathbb{R}^n, A)$ defines pseudodifferential operators on $\mathcal{B}^*(E)$; in fact, for $F \in CB^\infty(\mathbb{R}^n, A)$, $L_F(\varphi) = F \times_J \varphi$, $\varphi \in S^A(\mathbb{R}^n)$, is the pseudodifferential operator of symbol $F(x - J\xi)$.

Rieffel proves that L_F , $F \in CB^\infty(\mathbb{R}^n, A)$ is a continuous operator on E , ([7], Corollary 4.7) and that L_F is adjointable on E ([7], Proposition 4.2).

The Heisenberg group acts on $\mathcal{B}^*(E)$ by conjugation in the following way.

Given $V \in \mathcal{B}^*(E)$,

$$(z, \zeta, t) \longrightarrow E_{z, \zeta, t}^{-1} V E_{z, \zeta, t}, \quad (z, \zeta, t) \in \mathbb{R}^{2n} \times \mathbb{R},$$

where

$$E_{z, \zeta, t} f(x) = e^{it} e^{i\zeta x} f(x - z), \quad f \in S^A(\mathbb{R}^n).$$

It is easy to see that $V_{z, \zeta} = E_{z, \zeta, t}^{-1} V E_{z, \zeta, t}$ does not depend on $t \in \mathbb{R}$.

We say that V is *Heisenberg-smooth* if the map $(z, \zeta) \longrightarrow V_{z, \zeta}$ is C^∞ , and, if $z \longrightarrow V_z$ is C^∞ , where $V_z = V_{z, 0}$, we say that V is *translation-smooth*.

When we are dealing with the scalar case, $A = \mathbb{C}$, we have the remarkable characterization of Heisenberg-smooth operators in $\mathcal{B}^*(E)$ given by H. O. Cordes, [3]: these are the pseudodifferential operators whose symbols are in $CB^\infty(\mathbb{R}^{2n})$.

In section 3 we prove that if a skew-symmetric, $n \times n$, matrix is given and if the C^* -algebra A is such that a suitably stated generalization of Cordes' characterization can be proved, then any Heisenberg-smooth operator $T \in \mathcal{B}^*(E)$, which commutes with every pseudodifferential operator with symbol $G(x + J\xi)$, for some $G \in CB^\infty(\mathbb{R}^n, A)$, is also a pseudodifferential operator with symbol $F(x - J\xi)$, for

some $F \in CB^\infty(\mathbb{R}^n, A)$. This is a rephrasing of a conjecture stated by Rieffel for an arbitrary A at the end of Chapter 4 of [7] (the operators $G(x+JD)$ are those obtained from the right regular representation for his deformed product on $CB^\infty(\mathbb{R}^n, A)$). That Cordes' characterization implies Rieffel's conjecture has already been proved for the scalar case, [5]. The Schwartz kernel argument used in [5] has to be avoided here, in the more general case.

2 $a(x, D) \in \mathcal{B}^*(E)$.

Let us consider a pseudodifferential operator on E such that, if $\varphi \in S^A(\mathbb{R}^n)$,

$$a(x, D)\varphi(x) = \int e^{i(x-y)\xi} a(x, \xi) \varphi(y) dy d\xi,$$

for $a \in CB^\infty(\mathbb{R}^{2n}, A)$, where $dy = (2\pi)^{-\frac{n}{2}} dy$. As in the scalar case, we can see that $a(x, D)\varphi(x)$ is well defined for each $x \in \mathbb{R}^n$, if $\varphi \in S^A(\mathbb{R}^n)$.

An example of such an operator is given by Rieffel:

Given a function $F \in CB^\infty(\mathbb{R}^n, A)$,

$$L_F \varphi(x) = \int e^{i(x-y)\xi} F(x - J\xi) \varphi(y) dy d\xi.$$

The integrals considered here are *oscillatory integrals* ([7], Chapter 1).

Let us see next the fundamental ideas of a generalization of the Calderón-Vaillancourt Theorem for operators on E .

First, let us see that $a(x, D)(\varphi) \in E$, for $\varphi \in S^A(\mathbb{R}^n)$.

Considering $L^2(\mathbb{R}^n, A)$ as the set of all functions $f : \mathbb{R}^n \rightarrow A$ such that $\int \|f(x)\|^2 dx < \infty$, with the “almost everywhere” equivalence relation, where we consider the norm $\|\cdot\|_{L_2}$ (defined in the usual way), we can prove that $a(x, D)(\varphi) \in L^2(\mathbb{R}^n, A)$, as follows:

Using integration by parts and the equation

$$(i+x)^\alpha e^{ixy} = (i+D_y)^\alpha e^{ixy}, \quad \alpha = (1, \dots, 1), \quad \text{where } D_y = -i\partial_y, \quad (2.1)$$

we obtain

$$\begin{aligned} a(x, D)(\varphi)(x) &= \int e^{ix\xi} a(x, \xi) \hat{\varphi}(\xi) d\xi = \\ &= (i+x)^{-\alpha} \int [(i+D_\xi)^\alpha e^{ix\xi}] a(x, \xi) \hat{\varphi}(\xi) d\xi = \end{aligned}$$

$$= (i+x)^{-\alpha} \int e^{ix\xi} [(i-D_\xi)^\alpha a(x, \xi) \hat{\varphi}(\xi)] d\xi,$$

where $\hat{\varphi}(\xi) = \int e^{-i\xi y} \varphi(y) dy$. Since the last integral is bounded, $a(x, D)(\varphi) \in L^2(\mathbb{R}^n, A)$.

On the other hand, since $\|\cdot\|_2 \leq \|\cdot\|_{L2}$, we can prove that $L^2(\mathbb{R}^n, A) \subseteq E$, so that $a(x, D)(\phi) \in E$.

Theorem 2.1. *Let $a \in CB^\infty(\mathbb{R}^{2n}, A)$. Then, $a(x, D)$ is a bounded operator on E . In fact, $\|a(x, D)\| \leq l\pi(a)$, for $l \in \mathbb{R}^+$ independent of a , and $\pi(a) = \sup\{\|\partial_x^\beta \partial_\xi^\gamma a\|_\infty ; \beta, \gamma \leq \alpha = (1, 1, \dots, 1)\}$.*

PROOF. To begin with, let us consider the case when a has compact support. Denoting $a(x, D)$ by T , for $\varphi, \psi \in S^A(\mathbb{R}^n)$, we look at $\langle \psi, T\varphi \rangle$, which equals $\langle \hat{\psi}, \widehat{T\varphi} \rangle$. (Here we are dealing with the Fourier transform in $L^2(\mathbb{R}^n, A)$, which is “unitary” on E , in the sense that $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$, $f, g \in L^2(\mathbb{R}^n, A) \subseteq E$.)

Since

$$\widehat{T\varphi}(\eta) = \int e^{-i\eta x} T\varphi(x) dx = \int e^{-i\eta x} e^{i(x-y)\xi} a(x, \xi) \varphi(y) dy d\xi dx,$$

we have

$$\langle \hat{\psi}, \widehat{T\varphi} \rangle = \int e^{-i\eta x} e^{i(x-y)\xi} \hat{\psi}^*(\eta) a(x, \xi) \varphi(y) dy d\xi dx d\eta.$$

Using integration by parts and the equation (2.1), we have

$$\begin{aligned} \langle \hat{\psi}, \widehat{T\varphi} \rangle &= \int e^{-i\eta x} (i+x-y)^{-\alpha} \hat{\psi}^*(\eta) (i+x-y)^\alpha e^{i(x-y)\xi} a(x, \xi) \varphi(y) dy d\xi dx d\eta = \\ &= \int e^{-ix\eta} (i+x-y)^{-\alpha} \hat{\psi}^*(\eta) e^{ix\xi} e^{-iy\xi} [(i-D_\xi)^\alpha a(x, \xi)] \varphi(y) dy d\xi dx d\eta = \\ &= \int e^{-iy\xi} (i+x-y)^{-\alpha} (i+\xi-\eta)^{-\alpha} \hat{\psi}^*(\eta) e^{ix(\xi-\eta)} (i+\xi-\eta)^\alpha [(i-D_\xi)^\alpha a(x, \xi)] \varphi(y) dy d\xi dx d\eta = \\ &= \int e^{ix\xi} e^{-ix\eta} (i+x-y)^{-\alpha} \hat{\psi}^*(\eta) e^{-iy\xi} (i+\xi-\eta)^{-\alpha} [(i-D_x)^\alpha (i-D_\xi)^\alpha a(x, \xi)] \varphi(y) dy d\eta d\xi dx. \end{aligned}$$

Let us consider

$$h(z) = (i-z)^{-\alpha} \quad \alpha = (1, \dots, 1)$$

and

$$h_z(y) = h(y - z) \quad y, z \in \mathbb{R}^n.$$

Thus, we have

$$\langle \hat{\psi}, \widehat{T\varphi} \rangle = \int e^{ix\xi} f(x, \xi) [(i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi)] g(x, \xi) dx d\xi,$$

with

$$f(x, \xi) = \int e^{-ix\eta} h_\xi(\eta) \hat{\psi}(\eta)^* d\eta$$

and

$$g(x, \xi) = \int e^{-iy\xi} h_x(y) \varphi(y) dy.$$

So, we can write (by abuse of notation),

$$\| \langle \hat{\psi}, \widehat{T\varphi} \rangle \| = (2\pi)^{-\frac{n}{2}} \| \langle e^{-ix\xi} f^*(x, \xi), [(i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi)] g(x, \xi) \rangle \|.$$

If $c(x, \xi) = (i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi)$, there exists $d_1 \in \mathbb{R}^+$, not depending on a , such that $\sup_{(x, \xi) \in \mathbb{R}^{2n}} \|c(x, \xi)\| < d_1 \pi(a)$.

In Prop. 2.1 below, we prove that there exists $d_2 \in \mathbb{R}^+$, not depending on φ or g , such that $\|g\|_2 \leq d_2 \|\varphi\|_2$, so that we have

$$\|cg\|_2 \leq d_1 \pi(a) d_2 \|\varphi\|_2.$$

In a similar way as in Proposition 2.1, we get that

$$\|f^*\|_2 \leq d_2 \|\psi\|_2,$$

then, for $k = d_1 d_2^2 (2\pi)^{-\frac{n}{2}}$, we have, for all $\varphi, \psi \in S^A(\mathbb{R}^n)$,

$$\| \langle \psi, a(x, D) \varphi \rangle \| \leq k \pi(a) \|\varphi\|_2 \|\psi\|_2.$$

As for the general case, we consider the function $a_\varepsilon \in CB^\infty(\mathbb{R}^{2n}, A)$ given by $a_\varepsilon(x, \xi) = \phi(\varepsilon x, \varepsilon \xi) a(x, \xi)$, for $0 < \varepsilon \leq 1$, and where $\phi \in C_c^\infty(\mathbb{R}^{2n})$ is such that $\phi \equiv 1$ close to zero. As we just have seen, we have that $\| \langle \psi, a_\varepsilon(x, D) \varphi \rangle \| \leq k \pi(a_\varepsilon) \|\varphi\|_2 \|\psi\|_2$. Just doing some computations, we get that there is $m \in \mathbb{R}^+$, not depending on a or ε , such that $\pi(a_\varepsilon) \leq m \pi(a)$. Besides, it is not difficult to see that we have $\lim_{\varepsilon \rightarrow 0} \langle \psi, a_\varepsilon(x, D) \varphi \rangle = \langle \psi, a(x, D) \varphi \rangle$. Actually,

$$\| \langle \psi, a(x, D) \varphi \rangle \| \xrightarrow[\varepsilon \rightarrow 0]{} \| \langle \psi, a_\varepsilon(x, D) \varphi \rangle \| \leq k m \pi(a) \|\varphi\|_2 \|\psi\|_2.$$

Considering, now, $a(x, D)\varphi$ in place of ψ , since $a(x, D)\varphi \in L_2(\mathbb{R}^n, A)$, we have that

$$\| \langle a(x, D)\varphi, a(x, D)\varphi \rangle \| \leq l\pi(a) \|a(x, D)\varphi\|_2 \|\varphi\|_2, \quad l = km, \forall \varphi \in S^A(\mathbb{R}^n)$$

as before. So, there is $l \in \mathbb{R}^+$, not depending on a , such that $\|a(x, D)\| \leq l\pi(a)$. \blacksquare

Proposition 2.1. *Given $\varphi \in S^A(\mathbb{R}^n)$, let $g(x, \xi) = \int e^{-iy\xi} (i+x-y)^{-\alpha} \varphi(y) dy$, then there exists $d \in \mathbb{R}^+$, not depending neither on g nor on φ , such that $\|g\|_2 \leq d\|\varphi\|_2$.*

PROOF. Let h_x be as before, and put $g(x, \xi) = \int e^{-iy\xi} h_x(y) \varphi(y) dy = \widehat{h_x \varphi}(\xi)$. Then

$$\begin{aligned} \int g(x, \xi)^* g(x, \xi) dx d\xi &= \int \langle \widehat{h_x \varphi}, \widehat{h_x \varphi} \rangle dx = \int \langle h_x \varphi, h_x \varphi \rangle dx \\ &= \int \overline{h(x)} h(x) dx \int \varphi(\xi)^* \varphi(\xi) d\xi. \end{aligned}$$

If $d = \left(\int |h(x)|^2 dx \right)^{\frac{1}{2}}$, we have $\|g\|_2 \leq d\|\varphi\|_2$. \blacksquare

Note 2.1. If $a \in CB^\infty(\mathbb{R}^{2n}, A)$, we denote by $\mathcal{O}(a)$ the pseudodifferential operator whose symbol is a .

Proposition 2.2. *There exists $p \in CB^\infty(\mathbb{R}^{2n}, A)$ such that*

$$\langle \mathcal{O}(a)\varphi, \psi \rangle = \langle \varphi, \mathcal{O}(p)\psi \rangle \quad \forall \varphi, \psi \in S^A(\mathbb{R}^n). \quad (2.2)$$

PROOF. First we prove that $p(y, \xi) = \int e^{-ix\eta} a(y-z, \xi-\eta)^* dz d\eta$ belongs to $CB^\infty(\mathbb{R}^{2n}, A)$. As for this, we use strongly the definition of oscillatory integrals given in [7], where we consider for a while the corresponding Fréchet space $CB^\infty(\mathbb{R}^{2n}, A)$.

Then, applying proposition 1.6 of [7], we can begin working with a of compact support, for which we can work as Cordes in chapter 1 section 4 of [2].

To obtain the general case, we apply the Dominated Convergence Theorem. Please, see details at proposition 4.6 of [6]. \blacksquare

Remark 2.1. The application $\mathcal{O} : CB^\infty(\mathbb{R}^{2n}, A) \rightarrow \mathcal{B}^*(E)$, given by $a \mapsto a(x, D)$, is well defined and it is easy to see that it is injective.

Remark 2.2. As in the scalar case, [2], chapter 8, we see that a pseudodifferential operator is Heisenberg-smooth, because $\|a(x, D)\|$ depends just on a finite number of seminorms of $a \in CB^\infty(\mathbb{R}^{2n}, A)$. Besides, for $T = a(x, D)$, we have $\partial_z^\beta \partial_\zeta^\gamma T_{z, \zeta} = \mathcal{O}(\partial_z^\beta \partial_\zeta^\gamma a_{z, \zeta})$, where $a_{z, \zeta}(x, \xi) = a(x+z, \xi+\zeta)$, $\beta, \gamma \in \mathbb{N}^n$ (for proving these results, we just need to do some computations which we can check in proposition 4.7 of [6]).

Note 2.2. Let \mathcal{H} be the subset of $\mathcal{B}^*(E)$ formed by the Heisenberg-smooth operators. We have that $\mathcal{O} : CB^\infty(\mathbb{R}^{2n}, A) \rightarrow \mathcal{H}$ is a well defined, injective application. For $A = \mathbb{C}$, we have that \mathcal{O} is a bijection, [3].

3 Pseudodifferential operators that commute with R_G .

Let us consider here the right regular representation of $CB^\infty(\mathbb{R}^n, A)$ for the deformed product:

$$R_GF = F \times_J G.$$

Lemma 3.1. *If an operator $T \in \mathcal{B}^*(E)$ commutes with R_φ for all $\varphi \in S^A(\mathbb{R}^n)$, there exists a sequence F_k in E such that $F_k \times_J \varphi$ converges to $T(\varphi)$, for all $\varphi \in S^A(\mathbb{R}^n)$.*

PROOF. Let us find, first, a sequence, e_k , such that, for all $\varphi \in S^A(\mathbb{R}^n)$, $e_k \times_J \varphi \rightarrow \varphi$ (convergence in the $\|\cdot\|_2$ norm). Since A is separable, it has an approximate unit $(u_k)_{k \in \mathbb{N}}$. For each $k \in \mathbb{N}$, let us consider a C^∞ -function $\phi_k : \mathbb{R}^n \rightarrow A$, with support the set $\{x \in \mathbb{R}^n / \|x\| \leq \frac{1}{k}\}$ such that $\int \phi_k(x) dx = u_k$. Then, let $e_k = \mathcal{F}^{-1}(\phi_k)$, where \mathcal{F} is the Fourier transform on $S^A(\mathbb{R}^n)$ (for details, see proposition 2.5 of [6]).

Then, since R_φ is a continuous operator on E , letting $F_k = Te_k \in E$, $R_\varphi(F_k)$ is well defined and we have $F_k \times_J \varphi = R_\varphi Te_k = TR_\varphi e_k = T(e_k \times_J \varphi)$. Hence, since $e_k \times_J \varphi \rightarrow \varphi$, we have $F_k \times_J \varphi \rightarrow T\varphi$, for all $\varphi \in S^A(\mathbb{R}^n)$. ■

Proposition 3.1. *If T is an operator in $\mathcal{B}^*(E)$ which is such that $[T, R_\varphi] = 0 \ \forall \varphi \in S^A(\mathbb{R}^n)$, then $T_{z,\zeta} = T_{z-J\zeta,0}$.*

PROOF. Since R_φ is continuous, for any $F \in E$ and $\varphi \in S^A(\mathbb{R}^n)$, we may write $L_F(\varphi) = R_\varphi(F)$ thus defining L_F as an operator from $S^A(\mathbb{R}^n)$ to E .

It is easy to see that

$$E_{z,\zeta}^{-1} L_f E_{z,\zeta} = E_{z-J\zeta,0}^{-1} L_f E_{z-J\zeta,0},$$

for $f \in S^A(\mathbb{R}^n)$. Then, using that $E_{z,\zeta}$ leaves $S^A(\mathbb{R}^n)$ invariant (here we are writing $E_{z,\zeta}$ for $E_{z,\zeta,0}$, earlier defined), we get

$$E_{z,\zeta}^{-1} L_F E_{z,\zeta} = E_{z-J\zeta,0}^{-1} L_F E_{z-J\zeta,0}, \quad (3.1)$$

for $F \in E$, so we have that $(L_F)_{z,\zeta} = (L_F)_{z-J\zeta,0}$. By lemma 3.1, there is a sequence F_k in E such that, for all $\varphi \in S^A(\mathbb{R}^n)$, $\lim_{k \rightarrow \infty} (L_{F_k})_{z,\zeta}(\varphi) = T_{z,\zeta}(\varphi)$ (by equation (3.1)), so that $T_{z,\zeta} = T_{z-J\zeta,0}$. ■

Corollary 3.1. *If $T \in \mathcal{B}^*(E)$ is such as in proposition 3.1 and is translation-smooth, then T is Heisenberg-smooth.*

Lemma 3.2. *Given $a \in CB^\infty(\mathbb{R}^{2n}, A)$, let $b = \prod_{j=1}^n (1 + \partial_{y_j})^2 (1 + \partial_{\zeta_j})^2 a$, and $\gamma(x) = \prod_{j=1}^n f(x_j)$, with*

$$f(x_j) = \begin{cases} x_j e^{-x_j} & \text{if } x_j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then we have $a(x, \xi) = \int \gamma(-z)\gamma(-\zeta)b(x + z, \xi + \zeta)dzd\zeta$.

In the scalar case, $A = \mathbb{C}$, we can see the proof in [2], chapter 8, corollary 2.4. The same argument is valid for the general case.

Theorem 3.1. *Let A be a C^* -algebra for which the above defined application $\mathcal{O} : CB^\infty(\mathbb{R}^{2n}, A) \longrightarrow \mathcal{H}$ is a bijection. Then, given an operator $T \in \mathcal{B}^*(E)$, translation-smooth, that commutes with R_φ for all $\varphi \in S^A(\mathbb{R}^n)$, there exists a function F in $CB^\infty(\mathbb{R}^n, A)$ such that $T = L_F$.*

PROOF. Since \mathcal{O} is a bijection, and by corollary 3.1, there exists $a \in CB^\infty(\mathbb{R}^{2n}, A)$ such that $T = \mathcal{O}(a)$. As in lemma 3.2, let $b = \prod_{j=1}^n (1 + \partial_{x_j})^2 (1 + \partial_{\xi_j})^2 a$ and $B = \mathcal{O}(b)$. Note that $B_{z, \zeta} = \prod_{j=1}^n (1 + \partial_{z_j})^2 (1 + \partial_{\zeta_j})^2 T_{z, \zeta}$, see remark 2.2.

Since $TR_\varphi = R_\varphi T$, it is not difficult to see that $T_{z, \zeta} R_\varphi = R_\varphi T_{z, \zeta}$, for all $\varphi \in S^A(\mathbb{R}^n)$. Then, we have $[B, R_\varphi] = 0$ for all $\varphi \in S^A(\mathbb{R}^n)$. So, by proposition 3.1, $B_{z, \zeta} = B_{z - J\zeta, 0}$, so that $b(x + z, \xi + \zeta) = b(x + z - J\zeta, 0)$.

By lemma 3.2, we get $a(z, \zeta) = a(z - J\zeta, 0)$. Choosing $F(z) = a(z, 0)$, we have $T = L_F$, with $F \in CB^\infty(\mathbb{R}^n, A)$, as was to be proved. ■

Remark 3.1. We have just proved that a pseudodifferential operator that commutes with all operators $G(x + JD)$, ($G \in S^A(\mathbb{R}^n)$), where J is a fixed skew-symmetric matrix, is of the form $F(x - JD)$, $F \in S^A(\mathbb{R}^n)$.

Remark 3.2. If $A = \mathbb{C}$, since we have the Cordes' characterization [3], we can see that theorem 3.1 gives us another proof of the main result of [5], without needing to apply the Schwartz kernel.

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